

Directed Subset Feedback Vertex Set is Fixed-Parameter Tractable*

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March 3, 2013

Abstract

Given a graph G and an integer k , the FEEDBACK VERTEX SET (FVS) problem asks if there is a vertex set T of size at most k that hits all cycles in the graph. Bodlaender (WG '91) gave the first fixed-parameter algorithm for FVS in undirected graphs. The fixed-parameter tractability status of FVS in directed graphs was a long-standing open problem until Chen et al. (STOC '08) showed that it is fixed-parameter tractable by giving an $4^k k! n^{O(1)}$ algorithm. In the subset versions of this problems, we are given an additional subset S of vertices (resp. edges) and we want to hit all cycles passing through a vertex of S (resp. an edge of S). Indeed both the edge and vertex versions are known to be equivalent in the parameterized sense. Recently the SUBSET FEEDBACK VERTEX SET in undirected graphs was shown to be FPT by Cygan et al. (ICALP '11) and Kakimura et al. (SODA '12). We generalize the result of Chen et al. (STOC '08) by showing that SUBSET FEEDBACK VERTEX SET in directed graphs can be solved in time $2^{2^{O(k)}} n^{O(1)}$, i.e., FPT parameterized by size k of the solution. By our result, we complete the picture for feedback vertex set problems and their subset versions in undirected and directed graphs.

The technique of random sampling of important separators was used by Marx and Razgon (STOC '11) to show that UNDIRECTED MULTICUT is FPT and was generalized by Chitnis et al. (SODA '12) to directed graphs to show that DIRECTED MULTIWAY CUT is FPT. In this paper we give a general family of problems (which includes DIRECTED MULTIWAY CUT and DIRECTED SUBSET FEEDBACK VERTEX SET among others) for which we can do random sampling of important separators and obtain a set which is disjoint from a minimum solution and covers its “shadow”. We believe this general approach will be useful for showing the fixed-parameter tractability of other problems in directed graphs.

*A preliminary version appeared in ICALP 2012

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1 Introduction

The FEEDBACK VERTEX SET (FVS) problem has been one of the most extensively studied problems in the parameterized complexity community. Given a graph G and an integer k , it asks if there is a set T of size at most k which hits all cycles in G . FVS in both undirected and directed graphs was shown to be NP-hard by Karp [19]. A generalization of the FVS problem is the SUBSET FEEDBACK VERTEX SET (SFVS) problem: given a subset $S \subseteq V$ (resp. $S \subseteq E$), find a set T of size at most k such that T hits all cycles passing through a vertex of S (resp. an edge of S). It is easy to see that $S = V$ (resp. $S = E$) gives the FVS problem.

As compared to undirected graphs, FVS behaves differently on digraphs. In particular the trick of replacing each edge of an undirected graph G by arcs in both directions does not work: every feedback vertex set of the resulting digraph is a vertex cover of G and vice versa. Any other simple transformation does not seem possible either and thus the directed and undirected versions are very different problems. This is reflected in the best known approximation ratio for the directed versions as compared to the undirected problems: FVS in undirected graphs has an 2-approximation [1] while FVS in directed graphs has an $O(\log |V| \log \log |V|)$ -approximation [14, 29]. For SFVS in undirected graphs there is an 8-approximation [15] while the best-known approximation in directed graphs is $O(\min\{\log |V| \log \log |V|, \log^2 |S|\})$ [14].

Rather than finding approximate solutions in polynomial time, one can look for exact solutions in time that is superpolynomial, but still better than the running time obtained by brute force solutions. In both the directed and the undirected versions of the feedback vertex set problems, brute force can be used to check in time $n^{O(k)}$ if a solution of size at most k exists: one can go through all sets of size at most k . Thus the problem can be solved in polynomial time if the optimum is assumed to be small. In the undirected case, we can do significantly better: since the first FPT algorithm for FVS in undirected graphs by Bodlaender [3] almost 21 years ago, there have been a number of papers [2, 5, 6, 18] giving faster algorithms and the current fastest algorithm runs in $O^*(3^k)$ time [11] (the O^* notation hides all factors which are polynomial in size of input). That is, undirected FVS is fixed-parameter tractable parameterized by the size of the cutset we remove. Recall that a problem is *fixed-parameter tractable* (FPT) with a particular parameter p if it can be solved in time $f(p)n^{O(1)}$, where f is an arbitrary function depending only on p ; see [13, 16, 26] for more background. For digraphs, the fixed-parameter tractability status of FVS was a long-standing open problem (almost 16 years) until Chen et al. [8] resolved it by giving an $O^*(4^k k!)$ algorithm. This was recently generalized by Bonsma and Lokshantov [4] who gave a $O^*(47.5^k k!)$ algorithm for FVS in mixed graphs, i.e., graphs having both directed and undirected edges.

In the more general SUBSET FEEDBACK VERTEX SET problem, given an additional subset S of vertices and we want to find a set T of size at most k that hits all cycles passing through a vertex of S . In the edge version we are given a subset $S \subseteq E(G)$ and we want to hit all cycles passing through an edge of S . The vertex and edge versions are indeed known to be equivalent in the parameterized sense in both undirected and directed graphs. Recently Cygan et al. [12] and independently Kakimura et al. [17] have shown that SUBSET FEEDBACK VERTEX SET in undirected graphs is FPT parameterized by the size of the solution. Our main result is that SUBSET FEEDBACK VERTEX SET in digraphs is also fixed-parameter tractable parameterized by the size of the solution:

Theorem 1.1. (main result) SUBSET FEEDBACK VERTEX SET (SUBSET-DFVS) in directed graphs can be solved in $O^*(2^{2^{O(k)}})$ time.

Our techniques. As a first step, we use the standard technique of *iterative compression* [28] to argue that it is sufficient to solve the compression version of SUBSET-DFVS, where we assume that a solution T of size $k + 1$ is given in the input and we have to find a solution of size k . Our algorithm for the compression problem is inspired by the algorithm of Marx and Razgon [24] for undirected MULTICUT and Chitnis et al. [9] for DIRECTED MULTIWAY CUT. We define the “shadow” of a solution X as those vertices that are disconnected from T (in either direction) after the removal of X . Our goal is to ensure that there is a solution

whose shadow is empty, as finding such a shadowless solution can be a significantly easier task. For this purpose, we use the technique of “random sampling of important separators,” which was introduced in [24] for undirected graphs and was generalized to directed graphs in [9]. We present this approach here in generic way that can be used for the following general family of problems:

Finding an \mathcal{F} -transversal for some T -connected \mathcal{F}

Input : A directed graph $G = (V, E)$, a positive integer k , a set $T \subseteq V$ and a set $\mathcal{F} = \{F_1, F_2, \dots, F_q\}$ of subgraphs such that \mathcal{F} is T -connected, i.e., $\forall i \in [q]$ each vertex of F_i can reach some vertex of T by a walk completely contained in F_i and is reachable from some vertex of T by a walk completely contained in F_i .

Parameter : k

Question : Does there exist an \mathcal{F} -transversal $W \subseteq V$ with $|W| \leq k$, i.e., a set W such that $F_i \cap W \neq \emptyset$ for every $i \in [q]$?

It is easy to see that the above family includes DIRECTED MULTIWAY CUT (take T as the set of terminals and \mathcal{F} as the set of all walks between different terminals) and the compression version of SUBSET-DFVS (take T as the solution that we want to compress and \mathcal{F} as set of all S -closed-walks). For this family of problems, we can invoke the random sampling of important separators technique and obtain a set which is disjoint from a minimum solution and covers its shadow. Given such a set, we can use (some problem specific variant of) the “torso operation” to find an equivalent instance that has a shadowless solution. Therefore, we can focus on the simpler task of finding a shadowless solution. We believe this will be a useful opening step in the design of FPT algorithms for other transversal and cut problems on digraphs.

In the case of undirected MULTICUT [24], if there was a shadowless solution, then the problem could be reduced to an FPT problem called ALMOST 2SAT. In the case of DIRECTED MULTIWAY CUT [9], if there was a solution whose shadow is empty, then the problem could be reduced to the undirected version which was known to be FPT. For SUBSET-DFVS, the situation is a bit more complicated. As mentioned above, we first use the technique of iterative compression to reduce the problem to an instance where we are given a solution T and we want to find a disjoint solution of size at most k . We define the “shadows” with respect to the solution T that we want to compress whereas in [9], the shadows were defined with respect to the terminal set T . The “torso” operation we define in this paper is specific to the SUBSET-DFVS problem and differs from the one defined in [9]. Even after ensuring that there is a solution T' whose shadow is empty, we are not done unlike in [9]. We then analyze the structure of the graph $G \setminus T'$ and use “pushing” to branch on some important separators. Then for each branch, we need to do the whole process of random sampling of important separators to find a solution whose shadow is empty. This is followed again by branching on important separators. We then repeat the two-step process mentioned above until the budget k becomes zero.

2 Preliminaries

Observe, that a directed graphs contains no cycles if and only if it contains no closed-walks, for this reason throughout the article we use the term closed-walks, since it is sometimes easier to show a closed walk and avoid discussion whether it is a simple cycle or not. A feedback vertex set is a set of vertices that hits all the closed-walks of the graph.

Definition 2.1. (feedback vertex set) Let G be a directed graph. A set $T \subseteq V(G)$ is a feedback vertex set of G if $G \setminus T$ does not contain any closed-walks.

This gives rise to the DIRECTED FEEDBACK VERTEX SET (DFVS) problem where we are given a directed graph G and we want to find if G has a feedback vertex set of size at most k . DFVS was shown to be FPT by Chen et al. [8], closing a long-standing open problem in the parameterized complexity community.

In this paper we consider a generalization of the DFVS problem where given a set $S \subseteq V(G)$, we ask if there exists a vertex set of size $\leq k$ that hits all closed-walks passing through S .

SUBSET DIRECTED FEEDBACK VERTEX SET (SUBSET-DFVS)

Input : A directed graph $G = (V, E)$, a set $S \subseteq V(G)$ and a positive integer k .

Parameter : k

Question : Does there exist a set $T \subseteq V(G)$ with $|T| \leq k$ such that $G \setminus T$ has no closed walk containing a vertex of S ?

It is easy to see that SUBSET-DFVS is a generalization of DFVS by setting $S = V(G)$. We also define a variant of SUBSET-DFVS where the set S is a subset of edges. First we define a special type of closed-walks:

Definition 2.2. (*S-closed-walk*) Let $G = (V, E)$ be a digraph and $S \subseteq E(G)$. A closed walk (starting and ending at same vertex) C in G is said to be a S -closed-walk if it contains an edge from S .

EDGE SUBSET DIRECTED FEEDBACK VERTEX SET (EDGE-SUBSET-DFVS)

Input : A directed graph $G = (V, E)$, a set $S \subseteq E(G)$ and a positive integer k .

Parameter : k

Question : Does there exist a set $T \subseteq V(G)$ with $|T| \leq k$ such that $G \setminus T$ has no S -closed-walks?

The above two problems are shown to be equivalent in Appendix A.

2.1 Iterative Compression

We now use the technique of *iterative compression* introduced by Reed et al. [28]. It has been used to obtain faster FPT algorithms for various problems [6, 8, 24]. In the first step we transform the SUBSET-DFVS problem into the following problem:

SUBSET-DFVS REDUCTION

Input : A directed graph $G = (V, E)$, a set $S \subseteq E(G)$, a positive integer k and a set $T \subseteq V$ such that $G \setminus T$ has no S -closed-walks .

Parameter : $k + |T|$

Question : Does there exist a set $T' \subseteq V(G)$ with $|T'| \leq k$ such that $G \setminus T'$ has no S -closed-walks?

Lemma 2.3. (*power of iterative compression*) SUBSET-DFVS can be solved by $O(n)$ calls to an algorithm for the SUBSET-DFVS REDUCTION problem.

Proof. Denote $V = \{v_1, \dots, v_n\}$ and let $V_i = \{v_1, \dots, v_i\}$. We construct a sequence of subsets $X_i \subseteq V_i$, such that X_i is a solution for $G[V_i]$. Clearly, $X_1 = \emptyset$ is a solution for $G[V_1]$. Observe, that if X_i is a solution for $G[V_i]$, then $X_i \cup \{v_{i+1}\}$ is a solution for $G[V_{i+1}]$. Therefore, for each $i \in [n-1]$, we set $T = X_i \cup \{v_{i+1}\}$ and use, as a blackbox, an algorithm for SUBSET-DFVS REDUCTION, to construct a set X_{i+1} , which is a solution for $G[V_{i+1}]$, of size at most k . Note that if there is no solution for $G[V_i]$, for some $i \in [n]$, then there is no solution for the whole graph G and moreover, since $V_n = V$, if all the calls to the reduction problem are successful, X_n is a solution for the graph G . \square

Now we transform the SUBSET-DFVS REDUCTION problem into the following problem whose only difference is that the subset feedback vertex set in the output must be disjoint from the one in the input:

DISJOINT SUBSET-DFVS REDUCTION

Input : A directed graph $G = (V, E)$, a set $S \subseteq E(G)$, a positive integer k and a set $T \subseteq V$ such that $G \setminus T$ has no S -closed-walks.

Parameter : $k + |T|$

Question : Does there exist a set $T' \subseteq V(G)$ with $|T'| \leq k$ such that $T \cap T' = \emptyset$ and $G \setminus T'$ has no S -closed-walks?

Lemma 2.4. (*adding disjointness*) SUBSET-DFVS REDUCTION can be solved by $O(2^{|T|})$ calls to an algorithm for the DISJOINT SUBSET-DFVS REDUCTION problem.

Proof. Given an instance (G, S, T, k) of SUBSET-DFVS REDUCTION we guess the intersection I of T and the subset feedback vertex set T' in the output. We have at most $2^{|T|}$ choices for I . Then for each guess for I , we solve the DISJOINT SUBSET-DFVS REDUCTION problem for the instance $(G \setminus I, S, T \setminus I, k - |I|)$. It is easy to see that there is a solution T' for SUBSET-DFVS REDUCTION if and only if there is a guess I such that $T' \setminus I$ is returned by DISJOINT SUBSET-DFVS REDUCTION problem. \square

From Lemmas 2.3 and 2.4, an FPT algorithm for DISJOINT SUBSET-DFVS REDUCTION translates into an FPT algorithm for SUBSET-DFVS with an additional blowup factor of $O(2^{|T|}n)$.

3 Covering the Shadow of a Solution

The purpose of this section is to present the “random sampling of important separators” technique used in [9] for DIRECTED MULTIWAY CUT in a generalized way that applies to SUBSET-DFVS as well. The technique consists of two steps:

1. First find a set Z *small* enough to be disjoint from a solution X (of size $\leq k$) but *large* enough to cover the “shadow” of X .
2. Then define a “torso” operation which uses the set Z to reduce the problem instance in such a way that X becomes a shadowless solution.

In this section, we define a general family of problems for which Step 1 can be efficiently performed. The general technique to execute Step 1 is very similar to what was done for DIRECTED MULTIWAY CUT [9] and so we defer most of the proofs to the full version of the paper. In Section 4, we show how Step 2 can be done for the specific problem of DISJOINT SUBSET-DFVS REDUCTION. First we start by defining shadows:

Definition 3.1. (separator) Let $G = (V, E)$ be a directed graph. Given two disjoint non-empty sets $X, Y \subseteq V$ we call a set W of vertices as an $X - Y$ separator if W is disjoint from $X \cup Y$ and there is no walk from X to Y in $G \setminus W$. A set W is a minimal $X - Y$ separator if no proper subset of W is an $X - Y$ separator.

Definition 3.2. (shadow) Let G be graph and $W \subseteq V(G)$. Then for $v \in V(G)$ we say that v is in the “forward shadow” $f_{G,T}(W)$ of W (with respect to T), if W is a $T - \{v\}$ separator in G . Similarly, we say that v is in the “reverse shadow” $r_{G,T}(W)$ of W (with respect to T), if W is a $\{v\} - T$ separator in G .

That is, we can imagine T as a light source with light spreading on the directed edges. The forward shadow of W is the set of vertices that remain dark if the set W blocks the light. In the reverse shadow, we imagine that light is spreading on the edges backwards. We abuse the notation slightly and write $v - T$ separator instead of $\{v\} - T$ separator. We also drop G and T from the subscript if they are clear from the context. Note that W itself is not in the shadow of W (as a $T - v$ or $v - T$ separator needs to be disjoint from T and v), that is, W and $f_{G,T}(W) \cup r_{G,T}(W)$ are disjoint.

Let $G = (V, E)$ be a directed graph and $T \subseteq V(G)$. Consider $\mathcal{F} = \{F_1, F_2, \dots, F_q\}$ which is a set of subgraphs of G . We define the following property:

Definition 3.3. (*T*-connected) Let $\mathcal{F} = \{F_1, F_2, \dots, F_q\}$ be a set of subgraphs of G . Then \mathcal{F} is said to be *T*-connected if $\forall i \in [q]$, each vertex of the subgraph F_i can reach some vertex of T by a walk completely contained in F_i and is reachable from some vertex of T by a walk completely contained in F_i .

For a set \mathcal{F} of subgraphs of G , a transversal is a set of vertices which hits each subgraph in \mathcal{F} . We note that the subgraphs in \mathcal{F} are given implicitly to us.

Definition 3.4. (*F*-transversal) Let $\mathcal{F} = \{F_1, F_2, \dots, F_q\}$ be a set of subgraphs of G . Then W is said to be an *F*-transversal if $\forall i \in [q]$ we have $F_i \cap W \neq \emptyset$.

The main theorem of this section is the following:

Theorem 3.5. (randomized covering of the shadow) Let $T \subseteq V(G)$. In $O^*(4^k)$ time, we can construct a set $Z \subseteq V(G)$ such that for any set of subgraphs \mathcal{F} which is *T*-connected, if there exists an *F*-transversal of size $\leq k$, then the following holds with probability $2^{-2^{O(k)}}$: there is an *F*-transversal X of size $\leq k$ satisfying

1. $X \cap Z = \emptyset$.
2. Z covers the shadow of X .

The proof of Theorem 3.5 is given in Section 3.4. We also prove the following derandomized version of Theorem 3.5 in Section 3.5:

Theorem 3.6. (deterministic covering of the shadow) Let $T \subseteq V(G)$. In $O^*(2^{2^{O(k)}})$ time, we can construct a set $\{Z_1, Z_2, \dots, Z_t\}$ where $t = 2^{2^{O(k)}} \log^2 n$ such that for any set of subgraphs \mathcal{F} which is *T*-connected, if there exists an *F*-transversal of size $\leq k$, then there is an *F*-transversal X of size $\leq k$ such that for at least one $i \in [t]$ we have

1. $X \cap Z_i = \emptyset$.
2. Z_i covers the shadow of X .

In DIRECTED MULTIWAY CUT, T was the set of terminals and the set \mathcal{F} was the set of all walks from one vertex of T to another vertex of T . In SUBSET-DFVS, the set T is the solution that we want to compress and \mathcal{F} is the set of all closed S -walks passing through some vertex of T .

We say that an *F*-transversal T' is *shadowless* if $f(T') \cup r(T') = \emptyset$. Note that if T' is a shadowless solution, then in the graph $G \setminus T'$, each vertex is reachable from some vertex of T and can reach some vertex of T . In Section 5 we will see how we can make progress in DISJOINT SUBSET-DFVS REDUCTION if there exists a shadowless solution. So we would like to transform the instance in such a way that ensures the existence of a shadowless solution, by taking the torso (Section 4) and make progress by using the BRANCH algorithm from Section 5.

3.1 Important separators and random sampling

This subsection reviews the notion of important separators and the random sampling technique introduced in [24]. These ideas were later generalized for directed graphs in [9]. We closely follow [9] but give a self-contained presentation without relying on earlier work.

3.1.1 Important separators

Marx [23] introduced the concept of *important separators* to deal with the UNDIRECTED MULTIWAY CUT problem. Since then it has been used implicitly or explicitly in [7, 8, 9, 22, 24, 27] in the design of fixed-parameter algorithms. In this section, we define and use this concept in the setting of directed graphs. Roughly speaking, an important separator is a separator of small size that is *maximal* with respect to the set of vertices on one side.

Definition 3.7. (important separator) Let G be a directed graph and let $X, Y \subseteq V$ be two disjoint non-empty sets. A minimal $X - Y$ separator W is called an important $X - Y$ separator if there is no $X - Y$ separator W' with $|W'| \leq |W|$ and $R_{G \setminus W}^+(X) \subset R_{G \setminus W'}^+(X)$, where $R_A^+(X)$ is the set of vertices reachable from X in A .

In undirected graphs, an upper bound of 4^k on the number of important $X - Y$ separators of size at most k was given in [7] for any sets X, Y .

Lemma 3.8. $[\star]^1$ **(number of important separators)** Let $X, Y \subseteq V(G)$ be disjoint sets in a directed graph G . Then for every $k \geq 0$ there are at most 4^k important $X - Y$ separators of size at most k . Furthermore, we can enumerate all these separators in time $O^*(4^k)$.

For ease of notion, we define the following set of important separators:

Definition 3.9. (impsep) Given an instance (G, S, T, k) of DISJOINT SUBSET-DFVS REDUCTION, a set of vertices is called “impsep” if it is an important $v - T$ separator of size at most k in G for some vertex v in $V(G) \setminus T$.

It follows from Lemma 3.8 that the total number of impseps in an instance is at most $4^k \cdot |V(G)|$ and we can enumerate all of them in time $O^*(4^k)$. We now define a special type of shadows which we use later for the random sampling:

Definition 3.10. (exact shadow) Let (G, S, T, k) be an instance of DISJOINT SUBSET-DFVS REDUCTION. Let $W \subseteq V(G) \setminus T$ be a set of vertices. Then for $v \in V(G)$ we say that

1. v is in the “exact reverse shadow” of W (with respect to T), if W is a minimal $v - T$ separator in G , and
2. v is in the “exact forward shadow” of W (with respect to T), if W is a minimal $T - v$ separator in G .

The exact reverse shadow of W is a subset of the reverse shadow of W : roughly speaking, it contains a vertex v only if every vertex of W can be reached from v . This slight difference between the shadow and the exact shadow will be crucial in the analysis of the algorithm (Section 3.4).

The random sampling described in Section 3.1.2 (Theorem 3.14) randomly selects impseps and creates a subset by taking the union of the exact reverse shadows of the impseps. The following lemma will be used to give an upper bound on the probability that a vertex is covered by the union.

Lemma 3.11. Let z be any vertex. Then there are at most 4^k impseps in G which contain z in their exact reverse shadows.

For the proof of Lemma 3.11, we need to establish first the following:

Lemma 3.12. If W is an impsep and v is in the exact reverse shadow of W , then W is an important $v - T$ separator.

¹The proofs of the results labeled with \star have been deferred to the Appendix.

Proof. Let w be the witness that W is an impsep, i.e., W is an important $w - T$ separator in G . Let v be any vertex in the exact reverse shadow of W , which means that W is a minimal $v - T$ separator in G . Suppose that W is not an important $v - T$ separator. Then there exists a $v - T$ separator W' such that $|W'| \leq |W|$ and $R_{G \setminus W}^+(v) \subset R_{G \setminus W'}^+(v)$. We will arrive to a contradiction by showing that $R_{G \setminus W}^+(w) \subset R_{G \setminus W'}^+(w)$, i.e., W is not a important $w - T$ separator.

First, we claim that W' is an $(W \setminus W') - T$ separator. Suppose that there is a walk P from some $x \in W \setminus W'$ to T that is disjoint from W' . As W is a minimal $v - T$ separator, there is a walk Q from v to x whose internal vertices are disjoint from W . Furthermore, $R_{G \setminus W}^+(v) \subset R_{G \setminus W'}^+(v)$ implies that the internal vertices of Q are disjoint from W' as well. Therefore, concatenating Q and P gives a walk from v to T that is disjoint from W' , contradicting the fact that W' is a $v - T$ separator.

We show that W' is a $w - T$ separator and its existence contradicts the assumption that W is an important $w - T$ separator. First we show that W' is a $w - T$ separator. Suppose that there is a $w - T$ walk P disjoint from W' . The walk P has to go through a vertex $y \in W \setminus W'$ (as W is a $w - T$ separator). Thus by the previous claim, the subwalk of P from y to T has to contain a vertex of W' , a contradiction.

Finally, we show that $R_{G \setminus W}^+(w) \subseteq R_{G \setminus W'}^+(w)$. As $W \neq W'$ and $|W'| \leq |W|$, this will contradict the assumption that W is an important $w - T$ separator. Suppose that there is a vertex $z \in R_{G \setminus W}^+(w) \setminus R_{G \setminus W'}^+(w)$ and consider a $w - z$ walk that is fully contained in $R_{G \setminus W}^+(v)$, i.e., disjoint from W . As $z \notin R_{G \setminus W'}^+(v)$, walk Q contains a vertex $q \in W' \setminus W$. Since W' is a minimal $v - T$ separator, there is a $v - T$ walk that intersects W' only in q . Let P be the subwalk of this walk from q to T . If P contains a vertex $r \in W$, then the subwalk of P from r to T contains no vertex of W' (as $z \neq r$ is the only vertex of W' on P), contradicting our earlier claim that W' is a $(W \setminus W') - T$ separator. Thus P is disjoint from W , and hence the concatenation of the subwalk of Q from w to q and the walk P is a $w - T$ walk disjoint from W , a contradiction. \square

Lemma 3.11 easily follows from Lemma 3.12. Let J be an impsep such that z is in the exact reverse shadow of J . By Lemma 3.12, J is an important $z - T$ separator. By Lemma 3.8, there are at most 4^k important $z - T$ separators of size at most k and so z belongs to at most 4^k exact reverse shadows.

We note that this is the point where it is crucial to distinguish between “reverse shadow” and “exact reverse shadow”: Lemma 3.12 (and hence Lemma 3.11) does not remain true if we remove the word exact. Consider the following example. Let a_1, \dots, a_t be vertices such that there is an edge going from every a_i to every vertex of T . For every $1 \leq i \leq t$, let b_i be a vertex with an edge going from b_i to a_i . For every $1 \leq i < j \leq t$, let $c_{i,j}$ be a vertex with two edges going from $c_{i,j}$ to a_i and a_j . Then every set $\{a_i, a_j\}$ is an impsep, since it is an important $c_{i,j} - T$ separator. This means that every b_i is in the reverse shadow of $t - 1$ impseps. However, b_i is in the *exact* reverse shadow only of the impsep $\{a_i\}$.

3.1.2 Random sampling

The random sampling introduced in [24] was adapted to directed graphs in [9]. We follow it closely and try to present it in a self-contained way that might be useful for future applications.

In order to later reduce the problem (via the “torso” operation) to a shadowless instance, we need a set Z that has the following property:

There is a \mathcal{F} -transversal T^* such that Z covers the shadow of T^* , but Z is disjoint from T^* . (*)

Of course, when we are trying to construct this set Z , we do not know anything about the \mathcal{F} -transversals of the instance and in particular we have no way of checking if a given set Z satisfies this property. Nevertheless, we use a randomized procedure that creates a set Z and we give a lower bound on the probability that Z satisfies the requirements. For the construction of this set Z , we use a very specific probability distribution that was introduced in [24]. This probability distribution is based on randomly selecting “important separators” and taking the union of their shadows. The precise description of this function and the properties of

the distribution it creates is described in Theorem 3.14). The randomized selection can be derandomized: the randomized selection can be turned into a deterministic algorithm that returns a bounded number of sets such that at least one of them satisfies the required property (Section 3.2).

Roughly speaking, we want to select a random set Z such that for every (W, Y) where Y is in the reverse shadow of W , the probability that Z is disjoint from W but contains Y can be bounded from below. We can guarantee such a lower bound only if (W, Y) satisfies two conditions. First, it is not enough that Y is in the shadow of W (or in other words, W is an $Y - T$ separator), but W should contain important separators separating the vertices of Y from T (see Theorem 3.14 for the exact statement). Second, a vertex of W cannot be in the reverse shadow of other vertices of S , this is expressed by the following technical definition:

Definition 3.13. (thin) *Let G be a directed graph. We say that a set $W \subseteq V(G)$ is thin in G if there is no $v \in W$ such that v belongs to the reverse shadow of $W \setminus v$ with respect to T .*

Theorem 3.14. (random sampling) *There is an algorithm $\text{RandomSet}(G, T, k)$ that produces a random set $Z \subseteq V(G) \setminus T$ in time $O^*(4^k)$ such that the following holds. Let W be a thin set with $|W| \leq k$, and let Y be a set such that for every $v \in Y$ there is an important $v - T$ separator $W' \subseteq W$. For every such pair (W, Y) , the probability that the following two events both occur is at least $2^{-2^{O(k)}}$:*

1. $W \cap Z = \emptyset$, and
2. $Y \subseteq Z$.

Proof. The algorithm $\text{RandomSet}(G, T, k)$ first enumerates every impsep of size at most k ; let \mathcal{X} be the set of all exact reverse shadows of these impseps. By Lemma 3.8, the size of \mathcal{X} is $O^*(4^k)$ and can be constructed in time $O^*(4^k)$. Let \mathcal{X}' be the subset of \mathcal{X} where each element from \mathcal{X}' occurs with probability $\frac{1}{2}$ independently at random. Let Z be the union of the exact reverse shadows in \mathcal{X}' . We claim that the set Z satisfies the requirement of the theorem.

Let us fix a pair (W, Y) as in the statement of the theorem. Let $X_1, X_2, \dots, X_d \in \mathcal{X}$ be the exact reverse shadows of every impsep that is a subset of W . As $|W| \leq p$, we have $d \leq 2^k$. By assumption that W is *thin*, we have $X_j \cap W = \emptyset$ for every $j \in [d]$. Now consider the following events:

- (E1) $Z \cap W = \emptyset$
(E2) $X_j \subseteq Z \forall j \in [d]$

Note that (E2) implies that $Y \subseteq Z$. Our goal is to show that both events (E1) and (E2) occur with probability $2^{-2^{O(k)}}$.

Let $A = \{X_1, X_2, \dots, X_d\}$ and $B = \{X \in \mathcal{X} \mid X \cap W \neq \emptyset\}$. By Lemma 3.11, each vertex of W is contained in at most 4^k exact reverse shadows of impseps. Thus $|B| \leq |W| \cdot 4^k \leq k \cdot 4^k$. If no exact reverse shadow from B is selected, then event (E1) holds. If every exact reverse shadow from A is selected, then event (E2) holds. Thus the probability that both (E1) and (E2) occur is bounded from below by the probability of the event that every element from A is selected and no element from B is selected. Note that A and B are disjoint: A contains only sets disjoint from W , while B contains only sets intersecting W . Therefore, the two events are independent and the probability that both events occur is at least

$$\left(\frac{1}{2}\right)^{2^k} \left(1 - \frac{1}{2}\right)^{k \cdot 4^k} = 2^{-2^{O(k)}}$$

□

Algorithm 1: COVERING (randomized version)

Input: A directed graph G and a set T such that there is a T -connected set \mathcal{F} .

Output: A set Z .

- 1: Let $Z_1 = \text{RandomSet}(G, T, k)$.
 - 2: Let G_2 be obtained from G_1 by reversing the orientation of every edge and setting the weight of every vertex of Z_1 to infinity.
 - 3: Let $Z_2 = \text{RandomSet}(G_2, T, k)$.
 - 4: Let $Z = Z_1 \cup Z_2$.
-

3.2 Derandomization

We now derandomize the process of choosing exact reverse shadows in Theorem 3.14 using the technique of *splitters*. An (n, r, r^2) -splitter is a family of functions from $[n] \rightarrow [r^2]$ such that $\forall M \subseteq [n]$ with $|M| = r$, at least one of the functions in the family is injective on M . Naor, Schulman and Srinivasan [25] give an explicit construction of an (n, r, r^2) -splitter of size $O(r^6 \log(r) \log(n))$.

In the proof of Theorem 3.14, a random subset of a universe \mathcal{X} of size $n_0 = |\mathcal{X}| \leq 4^k n$ is selected. We argued that for a fixed S , there is a collection $A \subseteq \mathcal{X}$ of $a \leq 2^k$ sets and a collection $B \subseteq \mathcal{X}$ of $b \leq k \cdot 4^k$ sets such that if every set in A is selected and no set in B is selected, then events (E1) and (E2) hold. Instead of the selecting a random subset, we construct several subsets such that at least one of them satisfies both (E1) and (E2). Each subset is defined by a pair (h, H) , where h is a function in an $(n_0, a + b, (a + b)^2)$ -splitter family and H is a subset of $[(a + b)^2]$ of size a (there are $\binom{(a + b)^2}{a} = \binom{(2^k + k4^k)^2}{2^k} = 2^{2^{O(k)}}$ such sets H). For a particular choice of h and H , we select those exact shadows $W \in \mathcal{X}$ into \mathcal{X}' for which $h(W) \in H$. The size of the splitter family is $O((a + b)^6 \log(a + b) \log(n_0)) = 2^{O(k)} \log n$ and the number of possibilities for H is $2^{2^{O(k)}}$. Therefore, we construct $2^{2^{O(k)}} \log n$ subsets of \mathcal{X} .

By the definition of the splitter, there is a function h that is injective on $A \cup B$, and there is a subset H such that $h(L) \in H$ for every set L in A and $h(M) \notin H$ for every set M in B . For such an h and H , the selection will ensure that (E1) and (E2) hold. Thus at least one of the constructed subsets has the required properties, which we had to show.

3.3 The COVERING Algorithm

To prove Theorem 3.5, we show that Algorithm 1 gives a set Z satisfying the properties of Theorem 3.5. This is formalized in the following claim:

Claim 3.15. *The set Z in the output of Algorithm 1 satisfies the conditions of Theorem 3.5.*

Due to the delicate way separators behave in directed graphs, we construct the set Z in two phases, calling the function `RandomSet` twice. Our aim is to show that there is a solution T^* such that we can give a lower bound on the probability that Z_1 covers $r_{G,T}(T^*)$ and Z_2 covers $f_{G,T}(T^*)$. Note that the graph G_2 obtained in Step 2 depends on the set Z_1 returned in Step 1 (as we made the weight of every vertex in Z_1 infinite), thus the distribution of the second random sampling depends on the result Z_1 of the first random sampling. This means that we cannot make the two calls in parallel.

3.4 Proof of Theorem 3.5

For choosing T^* , we need the following definition:

Definition 3.16. (shadow-maximal solution)

A \mathcal{F} -transversal W is minimal if no proper subset of W is a solution. A minimal solution W is called shadow-maximal if $r_{G,T}(W) \cup f_{G,T}(W) \cup W$ is inclusion-wise maximal among all minimal solutions.

For the rest of the proof, let us fix T^* to be a shadow-maximal \mathcal{F} -transversal such that $|r_{G,T}(T^*)|$ is maximum possible among all shadow-maximal \mathcal{F} -transversals. We bound the probability that $Z \cap T^* = \emptyset$ and $r_{G,T}(T^*) \cup f_{G,T}(T^*) \subseteq Z$. More precisely, we bound the probability that all of the following four events occur:

1. $Z_1 \cap T^* = \emptyset$,
2. $r_{G,T}(T^*) \subseteq Z_1$,
3. $Z_2 \cap T^* = \emptyset$, and
4. $f_{G,T}(T^*) \subseteq Z_2$.

That is, the first random selection takes care of the reverse shadow, the second takes care of the forward shadow, and none of Z_1 or Z_2 hits T^* . Note that it is somewhat counterintuitive that we choose an T^* for which the shadow is large: intuitively, it seems that the larger the shadow is, the less likely that it is fully covered by Z . However, we need this maximality property in order to bound the probability that $Z \cap T^* = \emptyset$.

We want to invoke Theorem 3.14 to bound the probability that Z_1 covers $Y = r_{G,T}(T^*)$ and $Z_1 \cap T^* = \emptyset$. First, we need to ensure that T^* is a *thin* set, but this follows easily from the fact that T^* is a minimal \mathcal{F} -transversal:

Lemma 3.17. *If W is a minimal \mathcal{F} -transversal, then no $v \in W$ is in the reverse shadow of some $W' \subseteq W \setminus \{v\}$.*

Proof. Suppose to the contrary that there is a vertex $v \in W$ such that $v \in r(W')$ for some $W' \subseteq W \setminus \{v\}$. Then we claim that $W \setminus \{v\}$ is also a \mathcal{F} -transversal, contradicting the minimality of W . Let $\mathcal{F} = \{F_1, F_2, \dots, F_q\}$ and suppose $\exists i \in [q]$ such that $F_i \cap W = \{v\}$. As \mathcal{F} is T -connected, there is a $v \rightarrow T$ walk P in F_i . But $P \cap W = \{v\}$ implies that there is a $v \rightarrow T$ walk in $G \setminus (W \setminus \{v\})$, i.e., v cannot belong to the reverse shadow of any $W' \subseteq W \setminus \{v\}$. \square

More importantly, if we want to use Theorem 3.14 with $Y = r_{G,T}(T^*)$, then we have to make sure that for every vertex v of $r_{G,T}(T^*)$, there is an important $v - T$ separator that is a subset of T^* . The “pushing argument” of Lemma 3.18 shows that if this is not true for some v , then we can modify the \mathcal{F} -transversal in a way that increases the size of the reverse shadow. The extremal choice of T^* ensures that no such modification is possible, thus T^* contains an important $v - T$ separator for every v .

Lemma 3.18. (pushing) *Let W be a \mathcal{F} -transversal. For every $v \in r(W)$, either there is an $W_1 \subseteq W$ which is an important $v - T$ separator, or there is a \mathcal{F} -transversal W' such that*

1. $|W'| \leq |W|$,
2. $r(W) \subset r(W')$,
3. $(r(W) \cup f(W) \cup W) \subseteq (r(W') \cup f(W') \cup W')$.

Proof. Let W_0 be the subset of W reachable from v without going through any other vertices of W . Then W_0 is clearly a $v - T$ separator. Let W_1 be the minimal $v - T$ separator contained in W_0 . If W_1 is an important $v - T$ separator, then we are done as W itself contains W_1 . Otherwise, there exists an important $v - T$ separator W'_1 , i.e., $|W'_1| \leq |W_1|$ and $R_{G \setminus W_1}^+(v) \subset R_{G \setminus W'_1}^+(v)$. Now we show that $W' = (W \setminus W_1) \cup W'_1$ is also a \mathcal{F} -transversal. Note that $W'_1 \subseteq W'$ and $|W'| \leq |W|$.

First we claim that $r(W) \cup (W \setminus W') \subseteq r(W')$. Suppose that there is a walk P from β to T in $G \setminus W'$ for some $\beta \in r(W) \cup (W \setminus W')$. If $\beta \in r(W)$, then walk P has to go through a vertex $\beta' \in W$. As β' is

not in W' , it has to be in $W \setminus W'$. Therefore, by replacing β with β' , we can assume in the following that $\beta \in W \setminus W' \subseteq W_1 \setminus W'_1$. By minimality of W_1 , every vertex of $W_1 \subseteq W_0$ has an incoming edge from some vertex in $R_{G \setminus W}^+(v)$. This means that there is a vertex $\alpha \in R_{G \setminus W}^+(v)$ such that $(\alpha, \beta) \in E(G)$. Since $R_{G \setminus W}^+(v) \subset R_{G \setminus W'}^+(v)$, we have $\alpha \in R_{G \setminus W'}^+(v)$, implying that there is a $v \rightarrow \alpha$ walk in $G \setminus W'$. The edge $\alpha \rightarrow \beta$ also survives in $G \setminus W'$ as $\alpha \in R_{G \setminus W'}^+(v)$ and $\beta \in W_1 \setminus W'_1$. By assumption, we have a walk in $G \setminus W'$ from β to some $t \in T$. Concatenating the three walks we obtain a $v \rightarrow t$ walk in $G \setminus W'$ which contradicts the fact that W' contains an (important) $v - T$ separator W'_1 . Since $W \neq W'$ and $|W| = |W'|$, the set $W_1 \setminus W'_1$ is non-empty. Thus $r(W) \subset r(W')$ follows from the claim $r(W) \cup (W \setminus W') \subseteq r(W')$.

Suppose now that W' is not a \mathcal{F} -transversal. Then there is some $i \in [q]$ such that $F_i \cap W' = \emptyset$. W is a \mathcal{F} -transversal and so there is some $\beta \in W \setminus W'$ such that there is a $w \rightarrow T$ walk in F_i which gives a $w \rightarrow T$ walk in $G \setminus W'$ as $W' \cap F_i = \emptyset$. But we have $W \setminus W' \subseteq r(W')$ (by the claim in the previous paragraph), a contradiction. Thus W' is also a minimum \mathcal{F} -transversal.

Finally, we show that $r(W) \cup f(W) \cup W \subseteq r(W') \cup f(W') \cup W'$. We know that $r(W) \cup (W \setminus W') \subseteq r(W')$. Thus it is sufficient to consider a vertex $v \in f(W) \setminus r(W)$. Suppose that $v \notin f(W')$ and $v \notin r(W')$: there are walks P_1 and P_2 in $G \setminus W'$, going from T to v and from v to T , respectively. As $v \in f(W)$, walk P_1 intersects W , i.e., it goes through a vertex of $\beta \in W \setminus W' \subseteq r(W')$. However, concatenating the subwalk of P_1 from β to v and the walk P_2 gives a walk from $\beta \in r(W')$ to T in $G \setminus W'$, a contradiction. \square

Note that if W is a shadow-maximal \mathcal{F} -transversal, then the \mathcal{F} -transversal W' in Lemma 3.18 is also shadow-maximal. Therefore, by the extremal choice of T^* , applying Lemma 3.18 on T^* cannot produce a shadow-maximal \mathcal{F} -transversal T' with $r_{G,T}(T^*) \subset r_{G,T}(T')$, and hence T^* contains an important $v - T$ separator for every $v \in r_{G,T}(T^*)$. Thus by Theorem 3.14 for $Y = r_{G,T}(T^*)$, we get:

Lemma 3.19. *With probability at least $2^{-2^{O(k)}}$, both $r_{G,T}(T^*) \subseteq Z_1$ and $Z_1 \cap T^* = \emptyset$ occur.*

In the following, we assume that the events in Lemma 3.19 occur. Our next goal is to bound the probability that Z_2 covers $f_{G,T}(T^*)$. Note that T^* is a solution also of the instance (G_2, S, T, k) : the vertices in T^* remained finite (as $Z_1 \cap T^* = \emptyset$ by Lemma 3.19), and reversing the orientation of the edges does not change the fact that T^* is a solution. T^* is a shadow-maximal solution also in (G_2, S, T, p) : Definition 3.16 is insensitive to reversing the orientation of the edges and making some of the weights infinite can only decrease the set of potential solutions. Furthermore, the forward shadow of T^* in G_2 is same as the reverse shadow of T^* in G , that is, $f_{G_2,T}(T^*) = r_{G,T}(T^*)$. Therefore, assuming that the events in Lemma 3.19 occur, every vertex of $f_{G_2,T}(T^*)$ has infinite weight in G_2 . We show that now it holds that T^* contains an important $v - T$ separator in G_2 for every $v \in r_{G_2,T}(T^*) = f_{G,T}(T^*)$:

Lemma 3.20. *If W is a shadow-maximal \mathcal{F} -transversal and every vertex of $f(W)$ is infinite, then W contains an important $v - T$ separator for every $v \in r(W)$.*

Proof. Suppose to the contrary that there exists $v \in r(W)$ such that W does not contain an important $v - T$ separator. Then by Lemma 3.18, there is another shadow-maximal solution W' . As W is shadow-maximal, it follows that $r(W) \cup f(W) \cup W = r(W') \cup f(W') \cup W'$. Therefore, the nonempty set $W' \setminus W$ is fully contained in $r(W) \cup f(W) \cup W$. However it cannot contain any vertex of $f(W)$ (as they are infinite by assumption) and cannot contain any vertex of $r(W)$ (as $r(W) \subset r(W')$), a contradiction. \square

Recall that T^* is a shadow-maximal \mathcal{F} -transversal also in G_2 . In particular, T^* is a minimal \mathcal{F} -transversal for (G_2, S, T, k) and so by Lemma 3.17 we have that T^* is thin in G_2 also. Thus Theorem 3.14 can be used (with $Y = r_{G_2,T}(T^*)$) to bound the probability that $r_{G_2,T}(T^*) \subseteq Z_2$ and $Z_2 \cap T^* = \emptyset$. As the reverse shadow $r_{G_2,T}(T^*)$ in G_2 is the same as the forward shadow $f_{G,T}(T^*)$ in G , we can state the following:

Lemma 3.21. *Assuming the events in Lemma 3.19 occur, with probability at least $2^{-2^{O(k)}}$ both $f_{G,T}(T^*) \subseteq Z_2$ and $Z_2 \cap T^* = \emptyset$ occur.*

Therefore, with probability $(2^{-2^{O(k)}})^2$, the set $Z_1 \cup Z_2$ covers $f_{G,T}(T^*) \cup r_{G,T}(T^*)$ and it is disjoint from T^* . This proves Claim 3.15 and hence completes the proof of Theorem 3.5.

3.5 Proof of Theorem 3.6 via Derandomization of Algorithm 1

In Section 3.2, we have seen a deterministic variant of the function $\text{RandomSet}(G, T, k)$ that, instead of returning a random set Z , returns a deterministic set Z_1, \dots, Z_t of $t = 2^{2^{O(k)}} \log n$ sets. Therefore, in Steps 1 and 3 of Algorithm 1, we can replace RandomSet with this deterministic variant, and branch on the choice of one Z_i from the returned sets. By the properties of the deterministic algorithm, if I is a yes-instance, then Z has Property (*) in at least one of the $2^{2^{O(k)}} \log^2 n$ branches and therefore the algorithm finds a correct solution for I_1 . The branching increases the running time only by a factor of $(O^*(2^{2^{O(k)}}))^2$ and therefore the total running time is $O^*(2^{2^{O(k)}})$. This completes the proof of Theorem 3.6.

4 Reducing the Instance by Torso

We use the algorithm of Theorem 3.6 to construct a set Z of vertices that we want to get rid of. The second ingredient of our algorithm is an operation that removes a set of vertices without making the problem any easier. This transformation can be conveniently described using the operation of taking the *torso* of a graph. From this point onwards in the paper, we do not follow [9]. In particular, the *torso* operation is problem-specific. For DISJOINT SUBSET-DFVS REDUCTION, we define it as follows:

Definition 4.1. (torso) *Let (G, S, T, k) be an instance of DISJOINT SUBSET-DFVS REDUCTION and $C \subseteq V(G)$. The graph $\text{torso}(G, C)$ has vertex set C and there is (directed) edge (a, b) in $\text{torso}(G, C)$ if there is an $a \rightarrow b$ walk in G whose internal vertices are not in C . Furthermore, we add the edge (a, b) to S if there is an $a \rightarrow b$ walk in G which contains an edge from S and whose internal vertices are not in C .*

In particular, if $a, b \in C$ and (a, b) is a directed edge of G , then $\text{torso}(G, C)$ contains (a, b) as well. Thus $\text{torso}(G, C)$ is a supergraph of the subgraph of G induced by C . The following lemma shows that the *torso* operation preserves S -closed-walks inside C .

Lemma 4.2. (torso preserves S -closed-walks) *Let G be a directed graph and $C \subseteq V(G)$. Let $G' = \text{torso}(G, C)$, $v \in C$ and $W \subseteq C$. Then $G \setminus W$ has an S -closed-walk passing through v if and only if $G' \setminus W$ has an S -closed-walk passing through v .*

Proof. Let P be an S -closed-walk in $G \setminus W$ passing through v . If P is disjoint from W , then it contains vertices from C and $V(G) \setminus C$. Let u, w be two vertices of C such that every vertex of P between u and w is from $V(G) \setminus C$. Then, by definition of torso, there is an edge (u, w) in $\text{torso}(G, C)$. Using such edges we can modify P to obtain another closed walk say P' passing through v that lies completely in $\text{torso}(G, C)$ but avoids W . Note that since P is a S -closed-walk, at least one of the edges on some $u \rightarrow w$ walk that we short-circuited above must have been from S and by Definition 4.1 we would have put this (u, w) edge in S in the graph $G' = \text{torso}(G, C)$ which makes P' an S -closed-walk in G' .

Conversely suppose P' is an S -closed-walk passing through a vertex v in $\text{torso}(G, C)$ and it avoids $W \subseteq C$. If P' uses an edge $(u, w) \notin E(G)$, then this means that there is a $u \rightarrow w$ walk P'' whose internal vertices are not in C . Using such walks we modify P' to get a closed walk P_0 passing through v that only uses edges from G , i.e., P_0 is a closed walk in $G \setminus W$. It remains to show that P_0 is an S -closed-walk: since P' is an S -closed-walk, either some edge of P' was originally in S or there exist some $a, b \in P'$ such that there is a $a \rightarrow b$ walk does not contain any vertex from C and some edge on this walk was originally in S . \square

If we want to remove a set Z of vertices, then we create a new instance by taking the torso on the complement of Z :

Definition 4.3. Let $I = (G, S, T, k)$ be an instance of DISJOINT SUBSET-DFVS REDUCTION and $Z \subseteq V(G) \setminus T$. The reduced instance $I/Z = (G', S, T, p)$ is defined as

- $G' = \text{torso}(G, V(G) \setminus Z)$
- S is modified as specified in Definition 4.1.

The following lemma states that the operation of taking the torso does not make the DISJOINT SUBSET-DFVS REDUCTION problem easier for any $Z \subseteq V(G) \setminus T$ in the sense that any solution of the reduced instance I/Z is a solution of the original instance I . Moreover, if we perform the torso operation for a Z that is large enough to cover the shadow of some solution T^* and also small enough to be disjoint from T^* , then T^* becomes a shadowless solution for the reduced instance I/Z .

Lemma 4.4. (creating a shadowless instance) Let $I = (G, S, T, k)$ be an instance of DISJOINT SUBSET-DFVS REDUCTION and $Z \subseteq V(G) \setminus T$.

1. If I is a no-instance, then the reduced instance I/Z is also a no-instance.
2. If I has solution T' with $f_{G,T}(T') \cup r_{G,T}(T') \subseteq Z$ and $T' \cap Z = \emptyset$, then T' is a shadowless solution of I/Z .

Proof. Let G' be the graph $\text{torso}(G, V(G) \setminus Z)$ and let $C = V(G) \setminus Z$. To prove the first statement, suppose that $T' \subseteq V(G')$ is a solution for I/Z . We show that T' is also a solution for I . Suppose to the contrary that there exists a vertex $v \in T$ such that $G \setminus T'$ has an S -closed-walk passing through v (since $G \setminus T$ has no S -closed-walks). Note that $v \in T$ and $Z \subseteq V(G) \setminus T$ implies $v \in C$. Then by Lemma 4.2, $G' \setminus T'$ also has an S -closed-walks passing through v contradicting the fact that T' is a solution for I/Z .

For the second statement, let T' be a solution of I with $T' \cap Z = \emptyset$ and $f_{G,T}(T') \cup r_{G,T}(T') \subseteq Z$. We claim T' is a solution of I/Z as well. Suppose to the contrary that $G' \setminus T'$ has an S -closed-walk passing through some vertex $v \in C$. As $v \in C$, Lemma 4.2 implies $G \setminus T'$ also has an S -closed-walk passing through v , which is a contradiction as T' is a solution of I .

We claim that $r_{G',T}(T') = \emptyset$. Assume to the contrary that there exists $w \in r_{G',T}(T')$ (note that we have $w \in V(G')$, i.e., $w \notin Z$). So T' is a $w - T$ separator in G' , i.e., there is no $w - T$ walk in $G' \setminus T'$. Suppose there is a $w - T$ walk in $G \setminus T'$. Noting that $z \in C$ and $T \subseteq C$, when we take the torso to obtain G' this $w - T$ will be short-circuited but will be preserved in $G' \setminus T'$ which is a contradiction. Hence there is no $w - T$ walk in $G \setminus T'$, i.e., $w \in r_{G,T}(T')$. But $r_{G,T}(T') \subseteq Z$ and so we have $w \in Z$ which is a contradiction. Thus $r_{G,T}(T') \subseteq Z$ in G implies that $r_{G',T}(T')$ is empty in I/Z . The argument for $f_{G',T}(T') = \emptyset$ is analogous. \square

For every Z_i in the output of Theorem 3.6, we use the torso operation to remove the vertices in Z_i . We prove that this procedure is safe by showing the following:

Lemma 4.5. Let $I = (G, S, T, k)$ be an instance of DISJOINT SUBSET-DFVS REDUCTION. Let the sets in the output of Theorem 3.6 be Z_1, Z_2, \dots, Z_t . For every $i \in [t]$, let G_i be the reduced instance G/Z_i .

1. If I is a no-instance, then G_i is also a no-instance for every $i \in [t]$.
2. If I is a yes-instance, then there exists a solution T^* of I which is a shadowless solution of some G_j for some $j \in [t]$.

Proof. The first claim is easy to see: any solution T' of the reduced instance (G_i, S, T, k) is also a solution of (G, S, T, k) (by Lemma 4.4(1), the torso operation does not make the problem easier by creating new solutions).

By the derandomization of COVERING algorithm, there is a $j \in [t]$ such that Z has the Property $(*)$, i.e., there is a solution T^* of I such that $Z \cap T^* = \emptyset$ and Z covers shadow of T^* . Then Lemma 4.4(2) implies that T^* is a shadowless solution for the instance $G_j = I/Z_j$. \square

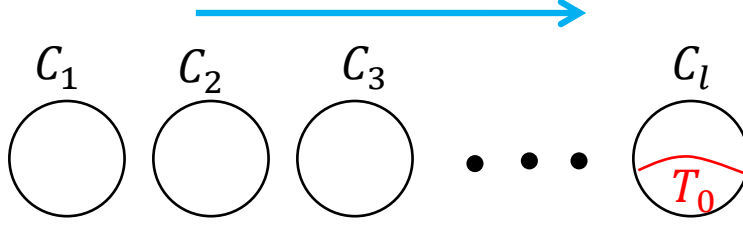


Figure 1: We arrange the strong components of $G \setminus T'$ in topological order so that the only possible direction of edges between the strong components is as shown by the blue arrow. We will claim later that the last component C_ℓ must contain a non-empty subset T_0 of T and further that no edge of S can be present within C_ℓ . This allows us to make some progress as we shall see in Theorem 5.4

5 Finding a Shadowless Solution

Consider an instance (G, S, T, k) of DISJOINT SUBSET-DFVS REDUCTION. First, let us assume that from each vertex of T , we can reach an edge of S , since otherwise we can clearly remove such a vertex from the set T , without violating the assumption that $G \setminus T$ has no S -closed walk. Next, we branch on all $2^{2^{O(k)}} \log^2 n$ choices for Z taken from $\{Z_1, Z_2, \dots, Z_t\}$ (given by Theorem 3.6) and build a reduced instance I/Z for each choice of Z . By Lemma 4.4, if I is a no-instance then I/Z_j is a no-instance for each $j \in [t]$. If I is a yes-instance, then by Lemma 4.5, there is at least one $i \in [t]$ such that I has a solution T' which is a solution, and in fact a shadowless solution, for the reduced instance I/Z_i .

So for the reduced instance I/Z_i we know that each vertex in $G \setminus T'$ can reach some vertex of T and can be reached from a vertex of T . Since T' is a solution for the instance (G, S, T, k) of DISJOINT SUBSET-DFVS REDUCTION, we know that $G \setminus T'$ does not have any S -closed-walks. Consider a topological ordering say C_1, C_2, \dots, C_ℓ of the strong components of $G \setminus T'$, i.e., there can be an edge from C_i to C_j only if $i < j$. We illustrate this in Figure 1.

Definition 5.1. (starting points of S) Let S^- be the set of starting points of edges in S , i.e., $S^- = \{u \mid (u, v) \in S\}$.

Lemma 5.2. (properties of C_ℓ) Let C_ℓ be the last strong component in the topological ordering of $G \setminus T'$ (refer to Figure 1). Then

1. C_ℓ contains a non-empty subset T_0 of T .
2. No edge of S is present within C_ℓ .
3. S^- is disjoint from C_ℓ .

Proof. We prove each of the three claims below:

1. If C_ℓ does not contain any vertex from T , then the vertices of C_ℓ cannot reach any vertex of T .
2. If C_ℓ contains an edge of S , then we will have an S -closed-walk in the strong component C_ℓ which is a contradiction as T' is a solution for the instance (G, S, T, k) of DISJOINT SUBSET-DFVS REDUCTION.
3. If a vertex say $v \in C_\ell \cap S^-$, then all outgoing edges from v must lie within C_ℓ . In particular, $v \in S^-$ implies there is a vertex w such that $(v, w) \in S$ which contradicts the second claim of the lemma.

□

Algorithm 2: BRANCH

Input: An instance $I = (G, S, T, k)$ of DISJOINT SUBSET-DFVS REDUCTION.

Output: A new set of $2^{O(k+|T|)}$ instances of DISJOINT SUBSET-DFVS REDUCTION where the budget k is reduced.

- 1: **for** every non-empty subset T_0 of T : **do**
 - 2: Use Lemma 3.8 to enumerate all the at most 4^k important $T_0 - (S^- \cup (T \setminus T_0))$ separators of size at most k .
 - 3: Let the important separators be $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$.
 - 4: **for** each $i \in [m]$ **do**
 - 5: Create a new instance $I_{T_0,i} = (G \setminus B_i, S, T, k - |B_i|)$ of DISJOINT SUBSET-DFVS REDUCTION.
-

Since T_0 is the subset of T present in C_ℓ and only edges between strong components can be from left to right, we have that there are no $T_0 - (T \setminus T_0)$ walks in $G \setminus T'$. Along with the third claim of Lemma 5.2, this implies that the solution T' contains a $T_0 - (S^- \cup (T \setminus T_0))$ separator. We now define a special type of separators:

Definition 5.3. (important separator) Let G be a digraph and let $X, Y \subseteq V$ be two disjoint non-empty sets. A minimal $X - Y$ separator W is called an important $X - Y$ separator if there is no $X - Y$ separator W' with $|W'| \leq |W|$ and $R_{G \setminus W}^+(X) \subset R_{G \setminus W'}^+(X)$, where $R_A^+(X)$ is the set of vertices reachable from X in A .

By “pushing”, we have the following theorem:

Theorem 5.4. (pushing) Either T' contains an important $T_0 - (S^- \cup (T \setminus T_0))$ separator or there is another solution T'' of the instance (G, S, T, k) such that $|T''| \leq |T'|$ and T'' contains an important $T_0 - (S^- \cup (T \setminus T_0))$ separator.

Proof. Let T^* be the subset of T' reachable from T_0 without going through any other vertices of T' . Then T^* is clearly a $T_0 - (S^- \cup (T \setminus T_0))$ separator. Let T^{**} be the minimal $T_0 - (S^- \cup (T \setminus T_0))$ separator contained in T^* . If T^{**} is an important $T_0 - (S^- \cup (T \setminus T_0))$ separator, then we are done as T' itself contains T^{**} .

Otherwise, there exists an important $T_0 - (S^- \cup (T \setminus T_0))$ separator T^{***} which dominates T^{**} , i.e., $|T^{***}| \leq |T^{**}|$ and $R_{G \setminus T^{**}}^+(T_0) \subset R_{G \setminus T^{***}}^+(T_0)$. Now we claim that $T'' = (T' \setminus T^{**}) \cup T^{***}$ is a solution for the instance (G, S, T, k) of DISJOINT SUBSET-DFVS REDUCTION. If we show this, then we are done as $|T''| \leq |T'|$ and T'' contains T^{***} which is an important $T_0 - (S^- \cup (T \setminus T_0))$ separator.

Suppose T'' is not a solution for the instance (G, S, T, k) of DISJOINT SUBSET-DFVS REDUCTION. However $|T''| \leq |T'| \leq k$ and $T'' \cap T = \emptyset$ as T'' contains an important $T_0 - (S^- \cup (T \setminus T_0))$ separator. Therefore this implies that there is an S -closed-walk in $G \setminus T''$ passing through some vertex $v \in T^{**} \setminus T^{***}$, i.e., there is a $v - S^-$ walk in $G \setminus T''$. Since T^{**} is a minimal $T_0 - (S^- \cup (T \setminus T_0))$ separator, we have $(T^{**} \setminus T^{***}) \subseteq R_{G \setminus T''}^+(T_0)$, i.e., $v \in R_{G \setminus T''}^+(T_0)$. This gives a $T_0 - S^-$ walk via v in $G \setminus T''$, a contradiction as T'' contains an (important) $T_0 - (S^- \cup (T \setminus T_0))$ separator. \square

Theorem 5.4 tells us that there is always a minimum solution which contains an important $T_0 - (S^- \cup (T \setminus T_0))$ separator where T_0 is a non-empty subset of T . This gives $2^{|T|} - 1$ choices for T_0 . For each guess of T_0 we enumerate all the at most 4^k important $T_0 - (S^- \cup (T \setminus T_0))$ separators of size at most k in time $O^*(4^k)$ as given by Lemma 3.8. This gives the following natural branching algorithm:

6 FPT Algorithm for DISJOINT SUBSET-DFVS REDUCTION

Lemma 4.5 and the BRANCH algorithm together combine to give a *bounded-search-tree* FPT algorithm for DISJOINT SUBSET-DFVS REDUCTION as follows:

FPT Algorithm for SUBSET-DFVS

Step 1: At the first step, for a given instance $I = (G, S, T, k)$, use Theorem 3.6 to obtain a set of instances $\{Z_1, Z_2, \dots, Z_t\}$ where $2^{2^{O(k)}} \log^2 n$ and Lemma 4.5 implies

- If I is a no-instance, then all the reduced instances $G_j = G/Z_j$ are no-instances for all $j \in [t]$
- If I is a yes-instance, then there is at least one $i \in [t]$ such that there is a solution T^* for I which is a shadowless solution for the reduced instance $G_i = G/Z_i$.

So at this step we branch into $2^{2^{O(k)}} \log^2 n$ directions.

Step 2 : For each of the instances obtained from the above step, we run the BRANCH algorithm to obtain a set of $2^{O(k+|T|)}$ instances where in each case either the answer is NO, or the budget k is reduced.

We then repeatedly perform Steps 1 and 2. Note that for every instance, one execution of steps 1 and 2 gives rise to $2^{2^{O(k)}} \log^2 n$ instances such that for each instance, either we know that the answer is NO or the budget k has decreased, because we have assumed that from each vertex of T one can reach the set S^- , and hence each important separator is non-empty. Therefore, considering a level as an execution of Step 1 followed by Step 2, the height of the search tree is at most k . Each time we branch into at most $2^{2^{O(k)}} \log^2 n$ directions (as $|T|$ is at most $k+1$). Hence the total number of nodes in the search tree is $\left(2^{2^{O(k)}} \log^2 n\right)^k$.

Lemma 6.1. *For every n and $k \leq n$, we have $(\log n)^k \leq (2k \log k)^k + \frac{n}{2^k}$*

Proof. If $\frac{\log n}{1+\log \log n} \geq k$, then $n \geq (2 \log n)^k$. Otherwise we have $\frac{\log n}{1+\log \log n} < k$ and then it is easy to verify that $(4k \log k)^k \geq (2 \log n)^k$. \square

So the total number of nodes in the search tree is $\left(2^{2^{O(k)}} \log^2 n\right)^k = \left(2^{2^{O(k)}}\right)^k (\log^2 n)^k = (2^{2^{O(k)}})(\log^2 n)^k \leq (2^{2^{O(k)}}) \left((2k \log k)^k + \frac{n}{2^k}\right)^2 \leq 2^{2^{O(k)}} n^2$. We then check the leaf nodes and see if there are any S -closed-walks left even after the budget k has become zero. If the graph at least one of the leaf nodes is S -closed-walk free, then the given instance is a yes-instance. Otherwise it is a no-instance. This gives an $O^*(2^{2^{O(k)}})$ algorithm for DISJOINT SUBSET-DFVS REDUCTION. By Lemma 2.3, we have an $O^*(2^{2^{O(k)}})$ algorithm for the SUBSET-DFVS problem.

7 Conclusion and Open Problems

In this paper we gave the first fixed-parameter algorithm for DIRECTED SUBSET FEEDBACK VERTEX SET parameterized by the size of the solution. Our algorithm used various tools from the FPT world such as iterative compression, bounded-depth search trees, random sampling of important separators, etc. We also gave a general family of problems for which we can do random sampling of important separators and obtain a set which is disjoint from a minimum solution and covers its shadow. We believe this general approach will be useful for deciding the fixed-parameter tractability status of other problems in digraphs where we do not know that much techniques unlike undirected graphs.

The next natural question is whether SUBSET-DFVS has a polynomial kernel or can we rule out such a possibility under some standard assumptions? The recent developments [10, 20, 21] in the field of kernelization may be useful in answering this question. Another question is to try and reduce the complexity of

our algorithm to single exponential. In the field of exact exponential algorithms, Razgon gave a $O(1.9977^n)$ algorithm for DFVS. It would be interesting to break the trivial $2^n n^{O(1)}$ barrier for SUBSET-DFVS.

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A Equivalence of SUBSET-DFVS and EDGE-SUBSET-DFVS

If (G, S, k) is an instance of SUBSET-DFVS we create an instance (G, S', k) of EDGE-SUBSET-DFVS by taking S' as the set of edges incident to any vertex of S . Then any closed walk passing through a vertex of S must pass through an edge of S' , and conversely any closed walk passing through an edge of S must contain a vertex from S .

On the other hand, given an instance (G, S', k) of EDGE-SUBSET-DFVS we create an instance (G', S, k) of SUBSET-DFVS where G' is obtained from G by the following modification: For every edge $(u, v) \in E(G)$ we add a new vertex x_{uv} and path $u \rightarrow x_{uv} \rightarrow v$ of length 2. We set $S = \{x_e : e \in S'\}$. Then any closed walk in G passing through an edge of S' corresponds to a closed-walk in G' which must pass through a vertex of S , and conversely any closed walk in G' passing through a vertex of S can be easily converted to a closed walk in G passing through an edge of S' . Both the reductions work in polynomial time and do not change the parameter. Therefore, in the rest of the paper we concentrate on solving the EDGE SUBSET DIRECTED FEEDBACK VERTEX SET problem and we shall refer to both the above problems as SUBSET-DFVS.

B Proof of Lemma 3.8

For the proof of Lemma 3.8, we first need to establish first some properties of separators.

Lemma B.1. *Let G be a directed graph and S be an important $X - Y$ separator. Then*

1. *For every $v \in S$, the set $S \setminus v$ is an important $X - Y$ separator in the graph $G \setminus v$.*
2. *If S is an $X' - Y$ separator for some $X' \supset X$, then S is also an important $X' - Y$ separator.*

Proof.

1. Suppose $S \setminus v$ is not a minimal $X - Y$ separator in $G \setminus v$. Let $S_0 \subset S \setminus v$ be a $X - Y$ separator in $G \setminus v$. Then $S_0 \cup v$ is a $X - Y$ separator in G but $S_0 \cup v \subset S$ which contradicts the fact that S is a minimal $X - Y$ separator in G . Now suppose $\exists S'$ such that $|S'| \leq |S \setminus v| = |S| - 1$ and $R_{(G \setminus v) \setminus (S \setminus v)}^+(X) \subset R_{(G \setminus v) \setminus S'}^+(X)$. But $R_{(G \setminus v) \setminus (S \setminus v)}^+(X) = R_{G \setminus S}^+(X)$ as deleting v from graph is equivalent to deleting it as part of the separator. Similarly $R_{(G \setminus v) \setminus S'}^+(X) = R_{G \setminus (S' \cup v)}^+(X)$. Therefore $R_{G \setminus S}^+(X) \subset R_{G \setminus (S' \cup v)}^+(X)$ and also $|S' \cup v| = |S'| + 1 \leq |S|$ which contradicts the fact that S is an important $X - Y$ separator.
2. Let S' be a witness that S is not important $X' - Y$ separator in G . Then $|S'| \leq |S|$ and S' is also an $X - Y$ separator. But S is important $X - Y$ separator and hence is also inclusion-wise minimal. Thus $S' \not\subseteq S$, i.e., $S' \setminus S \neq \emptyset$. Now the claim is $\exists s' \in S' \setminus S$ such that $s' \in R_{G \setminus S}^+(X)$. If we show this then $s' \in R_{G \setminus S}^+(X) \subseteq R_{G \setminus S}^+(X') \subset R_{G \setminus S'}^+(X')$ which is a contradiction as $s' \in S'$. So let us prove the claim. If any $X \rightarrow Y$ walk P contains a vertex of S' before reaching a vertex of S then we are done. So suppose every $X \rightarrow Y$ walk reaches S before S' . Then we have $R_{G \setminus S}^+(X) \subset R_{G \setminus S'}^+(X)$ which contradicts the fact that S is an important $X - Y$ separator as $|S'| \leq |S|$.

□

We need the following claim about submodularity of the function which is size of the out-neighborhood of a set. Recall that a function $f : 2^U \rightarrow \mathbb{N} \cup \{0\}$ is submodular if for all $A, B \subseteq U$ we have $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.

Lemma B.2. (submodularity) *Let $G = (V, E)$ be a directed graph. For $A \subseteq V$, let $N^+(A)$ be the out-neighborhood of set A , i.e., all the vertices in $G \setminus A$ which have an incoming edge from some vertex in A . Then the function $\gamma(A) = |N^+(A)|$ is submodular.*

Proof. Let $L = \gamma(A) + \gamma(B)$ and $R = \gamma(A \cup B) + \gamma(A \cap B)$. For any vertex $x \in V$ we have following four possibilities:

1. $x \notin N^+(A)$ and $x \notin N^+(B)$
In this case, x contributes 0 to both L and R .
2. $x \in N^+(A)$ and $x \notin N^+(B)$
In this case, x contributes 1 to L . Clearly $x \notin N^+(A \cap B)$. Also $x \in N^+(A \cup B)$ only if $x \notin B$ and so x can contribute at most 1 to R .
3. $x \notin N^+(A)$ and $x \in N^+(B)$
In this case, x contributes 1 to L . Clearly $x \notin N^+(A \cap B)$. Also $x \in N^+(A \cup B)$ only if $x \notin A$ and so x can contribute at most 1 to R .
4. $x \in N^+(A)$ and $x \in N^+(B)$
In this case, x contributes 2 to both L and R .

In all four cases the contribution of x to L is always greater equal its contribution to R and hence $L \geq R$, i.e., γ is submodular. □

We also require the following claim which gives a general family of $X - Y$ separators.

Lemma B.3. *If $X \subseteq Z$ and $Y \cap Z = \emptyset$, then the set $N^+(Z)$ is an $X - Y$ separator.*

Proof. We have $X \subseteq Z$ and $Z \cap Y = \emptyset$. By definition of N^+ , we have $N^+(Z)$ is $Z - Y$ separator and hence also a $X - Y$ separator. \square

Finally we are now ready to prove Lemma 3.8.:

Proof. To prove Lemma 3.8, we show by induction on $2k - \lambda$ that the number of important $X - Y$ separators of size at most k is upper bounded by $2^{2k - \lambda}$ where λ is the size of smallest $X - Y$ separator. Note that if $2k - \lambda < 0$, then $\lambda > 2k \geq k$ and so there is no (important) $X - Y$ separator of size at most k . If $2k - \lambda = 0$, then $\lambda = 2k$. Now if $k = 0$ then $\lambda = k = 0$ and the empty set is the unique important $X - Y$ separator of size at most k . If $k > 0$ then $\lambda = 2k > k$ and so there is no important $X - Y$ separator of size at most k . So we have checked the base case for induction. From now on, the induction hypothesis states that for any disjoint sets $X', Y' \subseteq V(G)$, any p such that $(2p - \beta) < (2k - \lambda)$ where β is the size of smallest $X' - Y'$ separator we have that the number of important $X' - Y'$ separators of size at most p is upper bounded by $2^{2p - \beta}$.

Recall that $R_{G \setminus S}^+(X)$ is the vertices reachable from X in $G \setminus S$. Now we prove a claim about uniqueness of minimum size separator whose “reach” is inclusion-wise maximal.

Lemma B.4. *There is a unique $X - Y$ separator S^* of size λ such that $R_{G \setminus S^*}^+(X)$ is inclusion-wise maximal.*

Proof. Suppose to the contrary that there are two separators S' and S'' of size λ such that $R_{G \setminus S'}^+(X)$ and $R_{G \setminus S''}^+(X)$ are incomparable and inclusion-wise maximal. By Lemma B.2, γ is submodular and hence

$$\gamma(R_{G \setminus S'}^+(X)) + \gamma(R_{G \setminus S''}^+(X)) \geq \gamma(R_{G \setminus S'}^+(X) \cup R_{G \setminus S''}^+(X)) + \gamma(R_{G \setminus S'}^+(X) \cap R_{G \setminus S''}^+(X))$$

By definition we have $\gamma(R_{G \setminus S'}^+(X)) = \lambda = \gamma(R_{G \setminus S''}^+(X))$. Let $Z = R_{G \setminus S'}^+(X) \cap R_{G \setminus S''}^+(X)$. Then $X \subseteq Z$ and $Z \cap Y = \emptyset$ as S', S'' are both $X - Y$ separators. By Lemma B.3, $N^+(R_{G \setminus S'}^+(X) \cap R_{G \setminus S''}^+(X))$ is a $X - Y$ separator and hence $\gamma(R_{G \setminus S'}^+(X) \cap R_{G \setminus S''}^+(X)) \geq \lambda$ which implies $\gamma(R_{G \setminus S'}^+(X) \cup R_{G \setminus S''}^+(X)) \leq \lambda$. Let $U = R_{G \setminus S'}^+(X) \cup R_{G \setminus S''}^+(X)$. By similar reasoning we have $X \subseteq U$ and $U \cap Y = \emptyset$. So $N^+(R_{G \setminus S'}^+(X) \cup R_{G \setminus S''}^+(X))$ is also a $X - Y$ separator. But we had $\gamma(R_{G \setminus S'}^+(X) \cup R_{G \setminus S''}^+(X)) \leq \lambda$ which implies $N^+(R_{G \setminus S'}^+(X) \cup R_{G \setminus S''}^+(X))$ is also a minimum separator which contradicts the maximality of $R_{G \setminus S'}^+(X)$ and $R_{G \setminus S''}^+(X)$. \square

Let S^* be the unique minimum separator given by Lemma B.4. The following claim shows that every important separator is “behind” this separator:

Lemma B.5. *For every important $X - Y$ separator S , we have $R_{G \setminus S^*}^+(X) \subseteq R_{G \setminus S}^+(X)$.*

Proof. Suppose this is not true for some S , then by submodularity of γ we have

$$\gamma(R_{G \setminus S^*}^+(X)) + \gamma(R_{G \setminus S}^+(X)) \geq \gamma(R_{G \setminus S^*}^+(X) \cup R_{G \setminus S}^+(X)) + \gamma(R_{G \setminus S^*}^+(X) \cap R_{G \setminus S}^+(X))$$

By definition, $\gamma(R_{G \setminus S^*}^+(X)) = \lambda$. As before $N^+(R_{G \setminus S^*}^+(X) \cap R_{G \setminus S}^+(X))$ is an $X - Y$ separator and hence $\gamma(R_{G \setminus S^*}^+(X) \cap R_{G \setminus S}^+(X)) \geq \lambda$. This implies $\gamma(R_{G \setminus S}^+(X)) \geq \gamma(R_{G \setminus S^*}^+(X) \cup R_{G \setminus S}^+(X))$ which contradicts the assumption that S is important $X - Y$ separator as $N^+(R_{G \setminus S^*}^+(X) \cup R_{G \setminus S}^+(X))$ is a $X - Y$ separator not larger than S but $R_{G \setminus S^*}^+(X) \cup R_{G \setminus S}^+(X)$ is a proper superset of $R_{G \setminus S}^+(X)$. Therefore, for every important separator S the set $R_{G \setminus S}^+(X)$ contains $R_{G \setminus S^*}^+(X)$. \square

Let $v \in S^*$ be an arbitrary vertex. Note that $\lambda > 0$ and so S^* is not empty. Any important $X - Y$ separator S of size at most k either contains v or not. If S contains v , then by Lemma B.1 (1), the set $S \setminus \{v\}$ is an important $X - Y$ separator in $G \setminus v$ of size at most $k' := k - 1$. As $v \notin X$, the size λ' of the minimum $X - Y$ separator in $G \setminus v$ is at least $\lambda - 1$. Therefore $2k' - \lambda' < 2k - \lambda$ and the induction hypothesis implies that there are at most $2^{2k' - \lambda'} \leq 2^{2k - \lambda - 1}$ important $X - Y$ separators of size k' in $G \setminus v$. Hence there are at most $2^{2k - \lambda - 1}$ important $X - Y$ separators of size at most k in G that contain v .

Now let us bound number of important $X - Y$ separators not containing v . By minimality of S^* , v has an in-neighbor in $R_{G \setminus S^*}^+(X)$. For every important $X - Y$ separator S , we have shown that $R_{G \setminus S^*}^+(X) \subseteq R_{G \setminus S}^+(X)$. As $v \notin S$ and v has an in-neighbor in $R_{G \setminus S^*}^+(X)$, even $R_{G \setminus S^*}^+(X) \cup \{v\} \subseteq R_{G \setminus S}^+(X)$ holds. Let $X' = R_{G \setminus S^*}^+(X) \cup \{v\}$. Then S is an $X' - Y$ separator as $R_{G \setminus S^*}^+(X) \cup \{v\} \subseteq R_{G \setminus S}^+(X)$. Since $X \subseteq X'$ and S is important $X - Y$ separator, by Lemma B.1 (2), S is in fact an important $X' - Y$ separator. Now there cannot exist an $X' - Y$ separator of size λ as such a set S would be a $X - Y$ separator of size λ in G as well with $R_{G \setminus S^*}^+(X) \cup \{v\} \subseteq R_{G \setminus S}^+(X)$ which contradicts the maximality of $R_{G \setminus S^*}^+(X)$. So the minimum size λ' of an $X' - Y$ separator in G is $> \lambda$. By the induction hypothesis, the number of important $X' - Y$ separators of size at most k in G is at most $2^{2k - \lambda'} \leq 2^{2k - \lambda - 1}$. Hence there are at most $2^{2k - \lambda - 1}$ important $X - Y$ separators of size at most k in G that do not contain v .

Adding the bounds in the two cases, we get the required bound of $2^{2k - \lambda}$.

An algorithm for enumerating all the at most 4^k important separators follows from the proof. First find a minimum $X - Y$ separator S' in polynomial time. Then for every $w \notin R_{G \setminus S'}^+(X)$, check in polynomial time if there is an $X - Y$ separator of size λ which does not contain any vertex from $R_{G \setminus S'}^+(X) \cup w$. This process will take us to the unique $X - Y$ separator S^* of size λ such that $R_{G \setminus S^*}^+(X)$ is inclusion-wise maximal. In the last step we branch on whether vertex $v \in S^*$ is in the important separator or not, and recursively find all possible important separators for both cases. \square